DIFFUSION MODEL OF LONGITUDINAL AGITATION IN HEAT AND MASS TRANSFER PROCESSES. PROBLEMS OF THE 2nd LEVEL

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The paper is a continuation of [1] on systematization of the employment of a diffusion model (DM) for describing heat and mass transfer processes. The object of consideration is a heat exchanger (a mass exchanging apparatus) with two flows moving in the DM regime, ideal displacement, and ideal agitation. All possible combinations of them are found. The problems are distributed according to three levels of complexity. The solution of problems of the 2nd level is considered.

In [1] criteria for systematization of heat and mass transfer processes on the basis of a diffusion model (DM) of longitudinal agitation of flows are suggested.

A heat exchanger (a mass exchanging apparatus) with two flows is the object of description. All possible combinations of flow structures are considered. Differential equations are written and boundary conditions are formulated for all the regions. An analysis is constructed on the basis of the carrying capacities of the transfer stages and the process as a whole. The criterion R is introduced, which is the ratio of the carrying capacity of the heat exchanger to the carrying capacity of transverse transfer. Particular descriptions are found and a general formulation of the calculation of R for the 1st-level problems is determined.

The problems of transfer are classified depending on the motion of heat carrier (phase) flows in the regimes of ideal displacement (ID) and ideal agitation (IA), which are limiting for the DM.

The concern of this paper is an analysis of the 2nd-level problems, which correspond to the motion of one flow in the DM regime and another in one of the limiting regimes (ID or IA), and the derivation of formulas for R and their generalization to this level.

All the quantities used are introduced in the first part of the work [1]. The numbering of formulas and tables is a continuation of [1].

We consider a technique for derivation of formulas for R of the 2nd-level models using the symbols introduced in [1] to identify the version of the motion scheme:

$$\begin{array}{l} \text{hot (DM)} \rightarrow \\ \text{cold (ID)} \leftarrow \end{array}.$$

For this case we take from Table 1 [1] the differential equations and boundary conditions

$$\ddot{T} - p\dot{T} - ap(T - t) = 0\,, \ \text{at} \ x = 0 \ \ T' = T - \frac{1}{p}\dot{T}\,, \ \text{at} \ x = 1 \ \ \dot{T} = 0\,;$$

$$\dot{t} + b(T - t) = 0\,, \ \text{at} \ x = 1 \ \ t' = t\,. \tag{16}$$

We solve the system by increasing the order of the differential equations for the temperature of the hot agent T

$$\dddot{T} - \ddot{T}(p + b) - \dot{T}(a - b)p = 0\,. \tag{17}$$

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The characteristic equation is

$$k^3 - (p + b)\, k^2 - (a - b)\, pk = 0\,,$$

$k_0 = 0$, and k_1, and k_2 are roots of the equation

$$k^2 - (p + b)\, k - (a - b)\, p = 0\,. \tag{18}$$

In subsequent solution the properties of the roots are useful:

$$k_1 k_2 = -\,(a - b)\, p\,,\ \ k_1 + k_2 = p + b\,.$$

The function

$$T = \lambda_0 + \lambda_1 \exp(k_1 x) + \lambda_2 \exp(k_2 x)\,. \tag{19}$$

is a solution of differential equation (17).

We substitute (19) into (16); then the temperature of a cold agent is

$$t = \lambda_0 + \lambda_1 \left(1 + \frac{k_1}{a} - \frac{k_1^2}{ap}\right) \exp(k_1 x) + \lambda_2 \left(1 + \frac{k_2}{a} - \frac{k_2^2}{ap}\right) \exp(k_2 x)\,.$$

Using the boundary conditions and denoting $T' - t' \equiv \Delta$, we write a system of equations for determining λ_1 and λ_2 (when the difference Δ in inlet temperatures is used there is no need to determine λ_0)

$$ap\Delta = \lambda_1 (ap - ak_1 - ap \exp k_1 - pk_1 \exp k_1 + k_1^2 \exp k_1) +$$

$$+ \lambda_2 (ap - ak_2 - ap \exp k_2 - pk_2 \exp k_2 + k_2^2 \exp k_2)\,;$$

$$0 = \lambda_1 k_1 \exp k_1 + \lambda_2 k_2 \exp k_2\,.$$

Solution of the system yields

$$\lambda_1 = \frac{ap\Delta k_2 \exp k_2}{k_2 \exp k_2 (ap - ak_1 + (-\,ap - pk_1 + k_1^2) \exp k_1) - k_1 \exp k_1 (ap - ak_2 + (-\,ap - pk_2 + k_2^2) \exp k_2)}\,,$$

$$\lambda_2 = \frac{-\,ap\Delta k_1 \exp k_1}{k_2 \exp k_2 (ap - ak_1 + (-\,ap - pk_1 + k_1^2) \exp k_1) - k_1 \exp k_1 (ap - ak_2 + (-\,ap - pk_2 + k_2^2) \exp k_2)}\,.$$

The heat flux transferred from the hot agent to the cold agent during heat transfer is

$$Q = G_1 C_1 (T' - T'') = G_1 C_1 \left(T_{x=0} - \frac{1}{p} \dot{T}_{x=0} - T_{x=1}\right) =$$

$$= G_1 C_1 \left(\lambda_1 \left(1 - \frac{k_1}{p} - \exp k_1\right) + \lambda_2 \left(1 - \frac{k_2}{p} - \exp k_2\right)\right).$$

Having divided both sides of the equation by $G_1 C_1 \Delta a$, we come to the criterion $R \equiv Q/(G_1 C_1 \Delta a)$. Having substituted λ_1 and λ_2 and transformed the equations (using the properties of the roots) we obtain as a result

$$R = \frac{(a - b + k_1) \exp(-\,k_2) - (a - b + k_2) \exp(-\,k_1) - (k_1 - k_2)}{a\,((a - b + k_1) \exp(-\,k_2) - (a - b + k_2) \exp(-\,k_1)) - b\,(k_1 - k_2)}$$

or (this is convenient for subsequent analysis)

$$R = \frac{\frac{1}{a-b}\left(1 - \frac{k_1 - k_2}{(a-b+k_1)\exp(-k_2) - (a-b+k_2)\exp(-k_1)}\right)}{1 + \frac{b}{a-b}\left(1 - \frac{k_1 - k_2}{(a-b+k_1)\exp(-k_2) - (a-b+k_2)\exp(-k_1)}\right)}. \tag{20}$$

The obtained equation for the 2nd level must turn into the corresponding equations of the 1st level given in Table 2 [1], depending on the limiting values of the Peclet number ($Pe \equiv p$): when $p \to 0$ (IA) and $p \to \infty$ (ID).

We show how this conversion takes place for $p \to \infty$

hot (DM) → ⇒ hot (ID) →
cold (ID) ← ⇒ cold (ID) ←.

We find the values of the roots of Eq. (18) for $p \to \infty$. For this purpose we express the discriminant $\sqrt{D}$ of the characteristic equation (18) as follows:

$$\sqrt{D} = \sqrt{(b+p)^2 - 4p(b-a)} = (p - b + 2a)\sqrt{\left(1 + \frac{b^2 - (-b+2a)^2}{(p-b+2a)}\right)}.$$

Hence $k_1 \to b-a$, $k_2 \to p$ for $p \to \infty$.

We substitute the values of $k_{1,2}$ into (20); after cancellations

$$R = \frac{1 - \exp(b-a)}{a - b\exp(b-a)},$$

or in final form

$$R = \frac{\frac{1}{a-b}(1 - \exp(-a+b))}{1 + \frac{b}{a-b}(1 - \exp(-a+b))},$$

which is in agreement with the earlier obtained equation (22) for a counterflow of agents in the ID regime in the 1st-level problem.

We now demonstrate the limiting transition for the case of $p \to 0$

hot (DM) → ⇒ hot (IA) ↔
cold (ID) ← ⇒ cold (ID) ←.

To determine to what the roots of Eq. (18) will tend, we write $\sqrt{D}$ so that

$$\sqrt{D} = (b-p)\sqrt{\left(1 + \frac{4ap}{(b-p)^2}\right)}.$$

Then, when $p \to 0$, the roots of the characteristic equation (18) are $k_1 \to 0$, $k_2 \to b$. After substitution into (20) we find

$$R = \frac{1 - \exp(-b)}{a(1 - \exp(-b)) + b},$$

or in final form

$$R = \frac{\frac{1}{b}(1 - \exp(-b))}{1 + \frac{a}{b}(1 - \exp(-b))},$$

which is in correspondence to the earlier obtained equation (13) for the considered 1st-level problem.

Thus, in both limiting cases, (20) naturally turns into the formulas derived earlier for the 1st level (this indirectly indicates the correctness of the 2nd-level formula).

We now consider the version of the 2nd-level problem when one of the agents moves in the IA regime

hot (DM) →

cold (IA) ↔.

According to Table 1 the differential equation has the form

$$\ddot{T} - p\dot{T} - ap(T - t'') = 0,$$

where $t'' = \text{const}$; the boundary conditions $T' = T - (1/p)\dot{T}$ at $x = 0$, $\dot{T} = 0$ at $x = 1$.

We replace the variable $y \equiv T - t''$; then

$$T = y + t'', \quad \dot{T} = \dot{y}, \quad \ddot{T} = \ddot{y}. \tag{21}$$

Hence

$$\ddot{y} - p\dot{y} - apy = 0. \tag{22}$$

The characteristic equation is

$$k^2 - pk - ap = 0. \tag{23}$$

The properties of the roots are

$$k_1 k_2 = -ap, \quad k_1 + k_2 = p.$$

A function of the form

$$y = \lambda_1 \exp(k_1 x) + \lambda_2 \exp(k_2 x). \tag{24}$$

is a solution of differential equation (22). Using the boundary conditions we obtain the system of equations

$$T' - t'' = \lambda_1 \left(1 - \frac{k_1}{p}\right) + \lambda_2 \left(1 - \frac{k_2}{p}\right),$$

$$0 = \lambda_1 k_1 \exp k_1 + \lambda_2 k_2 \exp k_2,$$

hence

$$\lambda_1 = \frac{(T' - t'')\, k_2 \exp k_2}{k_2 \left(1 - \frac{k_1}{p}\right) \exp k_2 - k_1 \left(1 - \frac{k_2}{p}\right) \exp k_1},$$

$$\lambda_2 = \frac{-(T' - t'')\, k_1 \exp k_1}{k_2 \left(1 - \frac{k_1}{p}\right) \exp k_2 - k_1 \left(1 - \frac{k_2}{p}\right) \exp k_1}.$$

We substitute λ_1 and λ_2 into Eq. (24)

$$y = (T' - t'')\frac{k_2 \exp(k_2 + k_1 x) - k_1 \exp(k_1 + k_2 x)}{k_2\left(1 - \frac{k_1}{p}\right)\exp k_2 - k_1\left(1 - \frac{k_2}{p}\right)\exp k_1} \tag{25}$$

and make a reverse substitution of variables according to Eq. (21).

The heat flux Q from the hot agent to the cold one is equal to

$$Q = G_1 C_1 (T' - T'') = G_2 C_2 (t'' - t'). \tag{26}$$

We express $T'' = T_{x=1}$ and substitute it into (26). Then with allowance for $T' - t' \equiv \Delta$

$$\Delta = Q\left(\frac{1}{G_1 C_1\left(1 - \frac{p(k_2 - k_1)}{k_2(p - k_1)\exp(-k_1) - k_1(p - k_2)\exp(-k_2)}\right)} + \frac{1}{G_2 C_2}\right),$$

whence, having introduced $Q/(\Delta K F) \equiv R$, we obtain

$$R = \frac{1}{\dfrac{a}{1 - \dfrac{p(k_2 - k_1)}{k_2(p - k_1)\exp(-k_1) - k_1(p - k_2)\exp(-k_2)}} + b},$$

or, after conversions,

$$R = \frac{\dfrac{1}{a}\left(1 - \dfrac{k_1 - k_2}{(a + k_1)\exp(-k_2) + (a + k_2)\exp(-k_1)}\right)}{1 + \dfrac{b}{a}\left(1 - \dfrac{k_1 - k_2}{(a + k_1)\exp(-k_2) - (a + k_2)\exp(-k_1)}\right)}. \tag{27}$$

The obtained equation of the 2nd level must turn into the corresponding equations of the 1st level given in Table 2 when $p \to 0$ (IA) and $p \to \infty$ (ID).

We show the limit transition for $p \to \infty$

hot (DM) → ⇒ hot (ID) →
cold (IA) ↔ ⇒ cold (IA) ↔ .

The values of the roots of Eq. (23) for $p \to \infty$ are

$$k_{1,2} = \frac{p \mp p\sqrt{\left(1 + 4\frac{a}{p}\right)}}{2}.$$

For further transformation we use the series-expansion formula $\sqrt{1 + \varepsilon} \approx (1 + \varepsilon/2)$ when $\varepsilon \ll 1$. Then

$$k_{1,2} \approx \frac{p \mp p\left(1 + 2\frac{a}{p}\right)}{2}$$

and $k_1 \to -a$, $k_2 \to p$.

We substitute the values of the roots in (27), then

$$R = \frac{1 - \exp(-a)}{a + b(1 - \exp(-a))}$$

or in final form

$$R = \frac{\frac{1}{a}(1 - \exp(-a))}{1 + \frac{b}{a}(1 - \exp(-a))},$$

which is in agreement with the earlier obtained equation (9) for the considered 1st-level problem.

We show that for $p \to 0$ the transition

$$\begin{matrix} \text{hot (DM)} \rightarrow \\ \text{cold (IA)} \leftrightarrow \end{matrix} \quad \Rightarrow \quad \begin{matrix} \text{hot (IA)} \leftrightarrow \\ \text{cold (IA)} \leftrightarrow \end{matrix}$$

is realized.

Solving Eq. (23)

$$k_{1,2} = \frac{p \pm \sqrt{4ap}\sqrt{\left(1 + \frac{p}{4a}\right)}}{2}$$

and using an expansion of the type $\sqrt{1 + \varepsilon} \approx (1 + \varepsilon/2)$ at small ε, we obtain $k_1 = -\sqrt{ap}$, $k_2 = \sqrt{ap}$.

We substitute these values of the roots into Eq. (27); as a result we have

$$R = \frac{\frac{1}{a}\left(1 - \frac{-2\sqrt{ap}}{(a - \sqrt{ap})\exp(-\sqrt{ap}) - (a + \sqrt{ap})\exp(\sqrt{ap})}\right)}{1 + \frac{b}{a}\left(1 - \frac{-2\sqrt{ap}}{(1 - \sqrt{ap})\exp(-\sqrt{ap}) - (a + \sqrt{ap})\exp(\sqrt{ap})}\right)}.$$

Expanding the functions $\exp(\sqrt{ap})$ and $\exp(-\sqrt{ap})$ into power series and restricting ourselves to the first two terms of the expansion (recall, we consider the case of $p \to 0$), we find

$$\exp(\sqrt{ap}) \approx 1 + (\sqrt{ap}), \quad \exp(-\sqrt{ap}) \approx 1 - \sqrt{ap}.$$

As a result

$$R = \frac{1}{1 + a + b},$$

which is in full agreement with Eq. (10) for the considered 1st-level problem.

In a similar way we can obtain the remaining formulas for the 2nd-level problems. We give all the twelve formulas of the 2nd level in combination with the corresponding characteristic equations (Table 3):

$$\begin{matrix} \text{hot (DM)} \rightarrow \\ \text{cold (ID)} \rightarrow \end{matrix} \quad k^2 - (p - b)k - (a + b)p = 0,$$

$$R = \frac{\frac{1}{a+b}\left(1 - \frac{k_1 - k_2}{(a + b + k_1)\exp(-k_2) - (a + b + k_2)\exp(-k_1)}\right)}{1 + \frac{0}{a+b}\left(1 - \frac{k_1 - k_2}{(a + b + k_1)\exp(-k_2) - (a + b + k_2)\exp(-k_1)}\right)}; \qquad (28)$$

hot (DM) → cold (IA) ↔ $\quad k^2 - pk - ap = 0\,,$

$$R = \frac{\frac{1}{a}\left(1 - \frac{k_1 - k_2}{(a + k_1)\exp(-k_2) - (a + k_2)\exp(-k_1)}\right)}{1 + \frac{b}{a}\left(1 - \frac{k_1 - k_2}{(a + k_1)\exp(-k_2) - (a + k_2)\exp(-k_1)}\right)}\,; \tag{29}$$

hot (DM) → cold (ID) ← $\quad k^2 - (p + b)\,k - (a - b)\,p = 0\,,$

$$R = \frac{\frac{1}{a-b}\left(1 - \frac{k_1 - k_2}{(a - b + k_1)\exp(-k_2) - (a - b + k_2)\exp(-k_1)}\right)}{1 + \frac{b}{a-b}\left(1 - \frac{k_1 - k_2}{(a - b + k_1)\exp(-k_2) - (a - b + k_2)\exp(-k_1)}\right)}\,; \tag{30}$$

hot (DM) ← cold (ID) → $\quad k^2 + (p + b)\,k - (a - b)\,p = 0\,,$

$$R = \frac{\frac{1}{a-b}\left(1 - \frac{-k_1 + k_2}{(a - b - k_1)\exp k_2 - (a - b - k_2)\exp k_1}\right)}{1 + \frac{b}{a-b}\left(1 - \frac{-k_1 + k_2}{(a - b - k_1)\exp k_2 - (a - b - k_2)\exp k_1}\right)}\,; \tag{31}$$

hot (DM) ← cold (IA) ↔ $\quad k^2 + pk - ap = 0\,,$

$$R = \frac{\frac{1}{a}\left(1 - \frac{-k_1 + k_2}{(a - k_1)\exp k_2 - (a - k_2)\exp k_1}\right)}{1 + \frac{b}{a}\left(1 - \frac{-k_1 + k_2}{(a - k_1)\exp k_2 - (a - k_2)\exp k_1}\right)}\,; \tag{32}$$

hot (DM) ← cold (ID) ← $\quad k^2 + (p - b)\,k - (a + b)\,p = 0\,,$

$$R = \frac{\frac{1}{a+b}\left(1 - \frac{-k_1 + k_2}{(a + b - k_1)\exp k_2 - (a + b - k_2)\exp k_1}\right)}{1 + \frac{0}{a+b}\left(1 - \frac{-k_1 + k_2}{(a + b - k_1)\exp k_2 - (a + b - k_2)\exp k_1}\right)}\,; \tag{33}$$

hot (ID) → cold (DM) → $\quad k^2 - (q - a)\,k - (a + b)\,q = 0\,,$

$$R = \frac{\frac{1}{a+b}\left(1 - \frac{k_1 - k_2}{(a + b + k_1)\exp(-k_2) - (a + b + k_2)\exp(-k_1)}\right)}{1 + \frac{0}{a+b}\left(1 - \frac{k_1 - k_2}{(a + b + k_1)\exp(-k_2) - (a + b + k_2)\exp(-k_1)}\right)}\,; \tag{34}$$

$$\begin{matrix}\text{hot (IA)} \leftrightarrow \\ \text{cold (DM)} \rightarrow\end{matrix} \quad k^2 - qk - bq = 0\,,$$

$$R = \frac{\dfrac{1}{b}\left(1 - \dfrac{k_1 - k_2}{(b + k_1)\exp(-k_2) - (b + k_2)\exp(-k_1)}\right)}{1 + \dfrac{a}{b}\left(1 - \dfrac{k_1 - k_2}{(b + k_1)\exp(-k_2) - (b + k_2)\exp(-k_1)}\right)}\,; \tag{35}$$

$$\begin{matrix}\text{hot (ID)} \leftarrow \\ \text{cold (DM)} \rightarrow\end{matrix} \quad k^2 - (q + a)\,k - (-a + b)\,q = 0\,,$$

$$R = \frac{\dfrac{1}{(-a + b)}\left(1 - \dfrac{k_1 - k_2}{(-a + b + k_1)\exp(-k_2) - (-a + b + k_2)\exp(-k_1)}\right)}{1 + \dfrac{a}{(-a + b)}\left(1 - \dfrac{k_1 - k_2}{(-a + b + k_1)\exp(-k_2) - (-a + b + k_2)\exp(-k_1)}\right)}\,; \tag{36}$$

$$\begin{matrix}\text{hot (ID)} \rightarrow \\ \text{cold (DM)} \leftarrow\end{matrix} \quad k^2 + (q + a)\,k - (-a + b)\,q = 0\,,$$

$$R = \frac{\dfrac{1}{(-a + b)}\left(1 - \dfrac{-k_1 + k_2}{(-a + b - k_1)\exp k_2 - (-a + b - k_2)\exp k_1}\right)}{1 + \dfrac{a}{(-a + b)}\left(1 - \dfrac{-k_1 + k_2}{(-a + b - k_1)\exp k_2 - (-a + b - k_2)\exp k_1}\right)}\,; \tag{37}$$

$$\begin{matrix}\text{hot (IA)} \leftrightarrow \\ \text{cold (DM)} \leftarrow\end{matrix} \quad k^2 + qk - bq = 0\,,$$

$$R = \frac{\dfrac{1}{b}\left(1 - \dfrac{-k_1 + k_2}{(b - k_1)\exp k_2 - (b - k_2)\exp k_1}\right)}{1 + \dfrac{a}{b}\left(1 - \dfrac{-k_1 + k_2}{(b - k_1)\exp k_2 - (b - k_2)\exp k_1}\right)}\,; \tag{38}$$

$$\begin{matrix}\text{hot (ID)} \leftarrow \\ \text{cold (DM)} \leftarrow\end{matrix} \quad k^2 + (q - a)\,k - (a + b)\,q = 0\,,$$

$$R = \frac{\dfrac{1}{a + b}\left(1 - \dfrac{-k_1 + k_2}{(a + b - k_1)\exp k_2 - (a + b - k_2)\exp k_1}\right)}{1 + \dfrac{0}{a + b}\left(1 - \dfrac{-k_1 + k_2}{(a + b - k_1)\exp k_2 - (a + b - k_2)\exp k_1}\right)}\,. \tag{39}$$

All these formulas of the 2nd level turn into the corresponding 1st-level equations for both hot ($p \to 0$, $p \to \infty$) and cold ($q \to 0$, $q \to \infty$) agents (this is an indirect confirmation of the correctness of the 2nd-level formulas).

Formulas (28)-(39) obtained for different 2nd-level problems contain criteria a and b in different combinations. For example, in the first three characteristic equations and formulas for R (29), (29), and (30) the difference is only in the multiplier l at b, which takes values of $+1$, -1, or 0. This makes it possible to increase the level of generalization of the characteristic equations and formulas for R.

We recall that the sign functions $\mathbf{l}$ (for the hot agent) and $\mathbf{m}$ (for the cold agent) are introduced in [1] to allow for the flow direction (see Table 2) and acquire the values: +1 for flow →, 0 for IA, and −1 for flow ←.

The presented formulas of the 2nd level were reduced to the following four expressions of a general form:

1) hot (DM) → (formulas (26)-(30)) to expressions

$$k^2 - (p - \mathbf{m}\, b)\, k - (a + \mathbf{m}\, b)\, p = 0\,, \tag{40}$$

$$R = \frac{\dfrac{1}{a + \mathbf{m}\, b}\left(1 - \dfrac{k_1 - k_2}{(a + \mathbf{m}\, b + k_1)\exp(-k_2) - (a + \mathbf{m}\, b + k_2)\exp(-k_1)}\right)}{1 + \dfrac{a + b - (a + \mathbf{m}\, b)}{(1 + |\mathbf{m}|)(a + \mathbf{m}\, b)}\left(1 - \dfrac{k_1 - k_2}{(a + \mathbf{m}\, b + k_1)\exp(-k_2) - (a + \mathbf{m}\, b + k_2)\exp(-k_1)}\right)}\,; \tag{41}$$

2) hot (DM) ← (formulas (31)-(33)) to expressions:

$$k^2 + (p + \mathbf{m}\, b)\, k - (a - \mathbf{m}\, b)\, p = 0\,, \tag{42}$$

$$R = \frac{\dfrac{1}{a - \mathbf{m}\, b}\left(1 - \dfrac{-k_1 + k_2}{(a - \mathbf{m}\, b - k_1)\exp k_2 - (a - \mathbf{m}\, b - k_2)\exp k_1}\right)}{1 + \dfrac{a + b - (a - \mathbf{m}\, b)}{(1 + |\mathbf{m}|)(a - \mathbf{m}\, b)}\left(1 - \dfrac{-k_1 + k_2}{(a - \mathbf{m}\, b - k_1)\exp k_2 - (a - \mathbf{m}\, b - k_2)\exp k_1}\right)}\,; \tag{43}$$

3) cold (DM) → (formulas (34)-(36)) to expressions:

$$k^2 - (q - \mathbf{l}a)\, k - (\mathbf{l}a + b)\, q = 0\,, \tag{44}$$

$$R = \frac{\dfrac{1}{\mathbf{l}\, a + b}\left(1 - \dfrac{k_1 - k_2}{(\mathbf{l}\, a + b + k_1)\exp(-k_2) - (\mathbf{l}\, a + b + k_2)\exp(-k_1)}\right)}{1 + \dfrac{a + b - (\mathbf{l}\, a + b)}{(1 + |\mathbf{l}|)(\mathbf{l}\, a + b)}\left(1 - \dfrac{k_1 - k_2}{(\mathbf{l}\, a + b + k_1)\exp(-k_2) - (\mathbf{l}\, a + b + k_2)\exp(-k_1)}\right)}\,; \tag{45}$$

4) cold (DM) ← (formulas (37)-(39)) to expressions:

$$k^2 + (q + \mathbf{l}\, a)\, k - (-\mathbf{l}\, a + b)\, q = 0\,, \tag{46}$$

$$R = \frac{\dfrac{1}{-\mathbf{l}\, a + b}\left(1 - \dfrac{-k_1 + k_2}{(-\mathbf{l}\, a + b - k_1)\exp k_2 - (-\mathbf{l}\, a + b - k_2)\exp k_1}\right)}{1 + \dfrac{a + b - (-\mathbf{l}\, a + b)}{(1 + |\mathbf{l}|)(-\mathbf{l}\, a + b)}\left(1 - \dfrac{-k_1 + k_2}{(-\mathbf{l}\, a + b - k_1)\exp k_2 - (-\mathbf{l}\, a + b - k_2)\exp k_1}\right)}\,. \tag{47}$$

A single formula generalizing all cases has not yet been obtained.

An analysis of general formulas (41), (43), (45), and (47) reveals situations when the calculation of R becomes impossible: in the case of the strict equality $a = b$ there arises division by zero. To overcome this difficulty we transform Eqs. (40) and (41). We introduce the notation $p - \mathbf{m}b \equiv f$, $a + \mathbf{m}b \equiv q$. Then (40) is rewritten as

$$k^2 - fk - gp = 0\,. \tag{48}$$

The roots of this quadratic equation are

TABLE 3. Levels of the Complexity of Problems and Numbers of Computational Formulas

Hot	Cold				
	ID	DM	IA	DM	ID
	→	→	↔	←	←
	m = +1	m = +1	m = 0	m = −1	m = −1
ID → l = +1	6*	34	9*	37	12*
DM → l = +1	28		29		30
IA ↔ l = −1	7*	35	10*	38	13*
DM ← l = −1	31		32		33
ID ← l = −1	8*	36	11*	39	14*

*Formulas (6)-(14) see in [1].

$$k_1 = \frac{1}{2} f \left(1 - \sqrt{\left(1 + \frac{4pg}{f^2}\right)}\right), \quad k_2 = \frac{1}{2} f \left(1 + \sqrt{\left(1 + \frac{4pg}{f^2}\right)}\right).$$

Expanding the radicals into series and restricting ourselves to the first two terms of the expansion (since $4pg/f^2 << 1$), we find the values of the roots

$$k_1 = -\frac{p}{f} g, \tag{49}$$

$$k_2 = f + \frac{p}{f} g. \tag{50}$$

To calculate R by formula (41) we should also expand into series the expressions exp $(-k_1)$ and exp $(-k_2)$

$$\exp(-k_1) \approx 1 - k_1 = 1 + \frac{p}{f} g. \tag{51}$$

To find an expression similar to exp $(-k_2)$ we use the properties of the roots of Eq. (48) $k_1 + k_2 = f$. Then exp $(-k_2)$ = exp $(-f)$ exp $k_1 = 1 + k_1$ (since $k_1 << 1$) and with account for (49) we have

$$\exp(-k_2) = \exp(-f)\left(1 - \frac{p}{f} g\right). \tag{52}$$

Now formula (41) can be presented in the form

$$R = \frac{\frac{1}{g}\left(1 - \frac{k_1 - k_2}{(g + k_1)\exp(-k_2) - (g + k_2)\exp(-k_1)}\right)}{1 + \frac{c}{g}\left(1 - \frac{k_1 - k_2}{(g + k_1)\exp(-k_2) - (g + k_2)\exp(-k_1)}\right)}, \tag{53}$$

where $c \equiv (a + b - (a + mb))/(1 + |m|)$ is a coefficient from Eq. (41).

We substitute expressions (49)-(52) into (53) and transform allowing for the fact that $g = 0$ is valid at $a = b$:

$$R = \frac{\frac{1}{f}\left(1 - \frac{p}{f}\right)(1 - \exp(-f)) + \frac{p}{f}}{1 + c\left[\left(1 - \frac{p}{f}\right)(1 - \exp(-f))\frac{1}{f} + \frac{p}{f}\right]}$$

or in final form

$$R = \frac{\left(\frac{1}{p - \mathrm{m}\, b}\right)\left(1 - \frac{p}{p - \mathrm{m}\, b}\right)(1 - \exp(-(p - \mathrm{m}\, b))) + \frac{p}{p - \mathrm{m}\, b}}{1 + c\left[\frac{1}{p - \mathrm{m}\, b}\left(1 - \frac{p}{p - \mathrm{m}\, b}\right)(1 - \exp(-(p - \mathrm{m}\, b))) + \frac{p}{p - \mathrm{m}\, b}\right]}. \qquad (54)$$

The derivation of similar formulas for the remaining general equations (43), (45), and (47) requires analogous transformations.

For convenience the numbers of all twelve obtained formulas are given in Table 3 according to the level of complexity and the sign variable.

REFERENCE

1. V. V. Zakharenko and T. N. Azyasskaya, Inzh.-Fiz. Zh., 70, No. 4, 688–694 (1997).